

# LOG-CONCAVITY OF ROWS OF PASCAL TYPE TRIANGLES

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**ABSTRACT.** Menon's proof of the preservation of log-concavity of sequences under convolution becomes simpler when adapted to 2-sided infinite sequences. Under assumption of log-concavity of two 2-sided infinite sequences, the existence of the convolution is characterised by a convergence criterion. Preservation of log-concavity under convolution yields a method of establishing the log-concavity of rows of a large class of Pascal type triangles, including a weighted generalization of the Delannoy triangle. This method is also compared with known techniques of proving log-concavity.

## 1. INTRODUCTION

For integer indexed 2-sided infinite sequences  $(a_n)_{n \in \mathbb{Z}}$  of non-negative real numbers we consider the *log-concavity* property without internal zeros, which means that for all integers  $n$  and for all positive integers  $p, q$  we have  $a_n a_{n+p+q} \leq a_{n+p} a_{n+q}$ . For the theory and applications of log-concavity of sequences, including further references, see e.g. [1, 5, 9, 13, 15, 16, 17].

**Fact 1.** *If  $(a_n)_{n \in \mathbb{Z}}$  and  $(b_n)_{n \in \mathbb{Z}}$  are log-concave then the term-wise product  $(a_n b_n)$  is also log-concave.*

As well known, every log-concave sequence is unimodal. Indeed the property of log-concavity itself can be characterized by the unimodality of certain derived sequences:

**Fact 2.** *The sequence  $(a_n)_{n \in \mathbb{Z}}$  is log-concave if and only if for each fixed  $p$  the skew self-product  $(a_n a_{p-n})_{n \in \mathbb{Z}}$  is unimodal.*

*Remark.* The modulus of  $(a_n a_{p-n})$  will be reached at  $\lfloor p/2 \rfloor$  (also at  $\lceil p/2 \rceil$ ), the floor and ceiling coinciding for even  $p$ .

**Fact 3.** *The following are equivalent for any non-null log-concave sequence  $(a_n)_{n \in \mathbb{Z}}$ :*

- (i) *the sequence has a finite sum,*
- (ii) *the terms of the sequence tend to 0 as  $n$  tends to infinity or minus infinity,*
- (iii) *the indices where the sequence has maximal values form a non-empty finite interval.*

**Proposition 1.** *Every term of the convolution of log-concave sequences  $(a_n)_{n \in \mathbb{Z}}$  and  $(b_n)_{n \in \mathbb{Z}}$  is finite if and only if for all  $p$  the terms of each skew product  $(a_n b_{p-n})$  tend to 0 as  $n$  tends to infinity or minus infinity.*

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## 2. PRESERVATION OF LOG-CONCAVITY UNDER CONVOLUTION

Certain properties of a sequence  $(a_n)_{n \in \mathbb{Z}}$ , not necessarily log-concave, are conveniently discussed in terms of the associated formal Laurent series  $\sum a_n X^n$ , the product of series  $(\sum a_n X^n)(\sum b_n X^n)$  corresponding to the *convolution sequence*  $(a_n) * (b_n)$  if it exists. For repeated convolutions of the sequence  $(a_n)_{n \in \mathbb{Z}}$  with itself ( $k$ -fold convolution with  $k$  factors) the exponential notation  $(a_n)^{*k}$  is used, where  $(a_n)^{*1} = (a_n)$  and  $(a_n)^{*0} = (\dots, 0, 0, 1, 0, 0, \dots)$ . An important class of Laurent series is that of power series, where  $a_n = 0$  for all negative indices.

It is known that the coefficient sequence of the product of two formal power series with log-concave non-negative coefficient sequences is also log-concave. This result was clearly stated and proved by Menon [14], and can also be deduced from the theory of total positivity of Karlin [7] developed in a larger context. In fact Karlin already deals with 2-sided infinite coefficient sequences, in other words, formal Laurent series. The product of two formal Laurent series with non-negative coefficients may not exist, but if it does, then log-concavity of the factors implies log-concavity of the product. This fact is somewhat simpler to verify for Laurent series than for power series (as Menon did), because Laurent series can be multiplied with any positive or negative power of the indeterminate  $X$ , a reversible operation which obviously preserves log-concavity of the coefficient sequence. Thus we need to verify only that in the product

$$(\sum a_n X^n)(\sum b_n X^n) = \sum c_n X^n$$

we have

$$c_0^2 \geq (c_{-1})(c_1). \quad (1)$$

The coefficients in question are infinite sums, assumed to be convergent,

$$\begin{aligned} c_0 &= \sum a_k b_{-k} \\ c_{-1} &= \sum a_k b_{-k-1} \\ c_1 &= \sum a_k b_{-k+1} \end{aligned}$$

For showing that (1) holds, the following general observation will be useful, applying to any situation when we need to compare two sums of non-negative real numbers indexed by the same index set, but term-by-term comparability for each index is not available. Recall that the sum of a countable family of non-negative real numbers is always a well-defined element of the extended real half-line  $[0, \infty]$ .

**Observation.** *Let  $(p_i)$  and  $(r_i)$  be finite or infinite families of non-negative real numbers indexed by the same set  $I$ , and let  $\sigma$  be a bijection  $I \rightarrow I$  (permutation of indices). If for each index  $i$  the inequality  $p_i + p_{\sigma(i)} \geq r_i + r_{\sigma(i)}$  holds, then  $\sum p_i \geq \sum r_i$  even if  $p_i \geq r_i$  fails for some indices  $i$ .*

Applying the Observation with  $I = \mathbb{Z}^2 = \{(k, n) : k, n \in \mathbb{Z}\}$ ,  $p_{k,n} = a_k a_n b_{-k} b_{-n}$  and  $r_{k,n} = a_k a_n b_{-k-1} b_{-n+1}$ , to show that  $c_0^2 \geq (c_{-1})(c_1)$  we need to verify that

$$\sum_{(k,n) \in \mathbb{Z}^2} p_{k,n} \geq \sum_{(k,n) \in \mathbb{Z}^2} r_{k,n} \quad (2)$$

For this, we define the permutation  $\sigma$  of the index set  $\mathbb{Z}^2$  by  $\sigma(k, n) = (n - 1, k + 1)$ . For each indexing pair  $(k, n)$ , we claim that

$$p_{k,n} + p_{\sigma(k,n)} \geq r_{k,n} + r_{\sigma(k,n)}$$

This inequality is equivalent to the non-negativity of

$$\begin{aligned} & a_k a_n b_{-k} b_{-n} + a_{n-1} a_{k+1} b_{-n+1} b_{-k-1} - a_k a_n b_{-k-1} b_{-n+1} - a_{n-1} a_{k+1} b_{-n} b_{-k} \\ &= a_k a_n (b_{-k} b_{-n} - b_{-k-1} b_{-n+1}) - a_{n-1} a_{k+1} (b_{-n} b_{-k} - b_{-n+1} b_{-k-1}) \\ &= (a_k a_n - a_{n-1} a_{k+1}) (b_{-k} b_{-n} - b_{-k-1} b_{-n+1}) \end{aligned}$$

Simple case analysis shows that the two parentheses above are of the same sign, based on the log-concavity of  $(a_n)$  and  $(b_n)$ . This argument, amounting to a version of Menon's proof for 1-sided infinite sequences [14], confirms the following.

**Proposition 2** (see e.g. Karlin[7], Theorem 8.2). *If every term of the convolution of log-concave sequences  $(a_n)_{n \in \mathbb{Z}}$  and  $(b_n)_{n \in \mathbb{Z}}$  is finite, then the convolution  $(a_n) * (b_n)$  is also log-concave.*

Note that the dual property of log-convexity is not preserved under convolution (see Liu and Wang [10]).

### 3. PASCAL TYPE TRIANGULAR ARRAYS

Several broad generalizations of Pascal's triangle were developed, in particular as an approach to log-concavity (or unimodality) of (finite) combinatorial sequences (e.g. Stirling numbers [15],  $f$ -vectors of polytopes). Kurtz's construction involves row to next row recursion using weighted summation of the two entries above a given entry [8], a simpler instance of which was used to study  $f$ -vectors of cyclic polytopes [12]. The triangular construction may also proceed from fixing the (infinite) right side of the triangle not necessarily consisting of 1's. This approach is implicit e.g. in Hoggar [5], the construction being in fact subsumed by the earlier convolution triangle construction of Hoggatt and Bicknell [6]. In this section we establish the log-concavity of the rows of a large class of convolution triangles, a special case being the log-concavity of the rows of the Delannoy triangle.

A *triangular array* is a doubly indexed family of numbers  $T(n, k)_{n, k \in \mathbb{Z}}$ , where  $T(n, k)$  is 0 unless  $0 \leq k \leq n$ . The  $n^{\text{th}}$  row of the array is the sequence  $(T(n, k))_{k \in \mathbb{Z}}$ . Obviously this row has at most  $n + 1$  non-null terms. For any two sequences  $a = (a_n)$  and  $q = (q_n)$  whose terms are zeros for  $n < 0$ , the *convolution array* is the triangular array given by  $T(n, k) = (a * q^{*(n-k)})_k$  for  $0 \leq k \leq n$ ,  $a = (a_n)$  being called *initial side sequence* and  $q = (q_n)$  the *convolution multiplier sequence*.

**Lemma.** *Let  $a = (a_n)$  and  $q = (q_n)$  be non-negative log-concave sequences whose terms are zeros for  $n < 0$ . For  $k \geq 1$  define the sequences*

$$b = a * q^{*(k-1)}$$

$$c = a * q^{*k}$$

$$d = a * q^{*(k+1)}$$

*For all  $n \geq 1$  the following inequality holds:*

$$c_n^2 \geq d_{n-1} \cdot b_{n+1}$$

*Proof.* Now

$$c_n^2 = (b_0 q_n + \dots + b_n q_0)(b_0 q_n + \dots + b_n q_0) \quad (3)$$

and

$$d_{n-1} \cdot b_{n+1} = \left( \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} b_i q_j q_{n-i-j-1} \right) \cdot b_{n+1} \quad (4)$$

From (3) we have

$$c_n^2 \geq \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} b_{i+j+1} b_{n-j} q_{n-i-j-1} q_j \quad (5)$$

while because of log-concavity the product  $b_{i+j+1} b_{n-j} q_{n-i-j-1} q_j$  in (5) is always greater than or equal to  $b_i q_j q_{n-i-j-1} b_{n+1}$ , which proves the Lemma due to the expression (4).  $\square$

Based on the Lemma the following can be verified:

**Proposition 3.** *For any two given log-concave sequences  $a = (a_n)$  and  $q = (q_n)$  whose terms are zeros for  $n < 0$ , taken as initial side and convolution multiplier sequence respectively, every row of the corresponding convolution array is log-concave.*

As an application, consider the triangular array which for any fixed real  $b, c, d \geq 0$  is given by  $T(0, 0) = 1$  and

$$T(n, k) = bT(n-1, k-1) + cT(n-1, k) + dT(n-2, k-1)$$

In other words, each entry in the array other than  $T(0, 0)$  is calculated as a weighted sum of the two entries above and the third entry which is above the latter two. This is in fact a weighted generalization of the Delannoy triangle (counting weighted lattice paths as in Fray and Roselle [3] and Heteyi [4], but being less general than the generalized Delannoy numbers in Dziemiańczuk [2], differing also from Loehr and Savage [11]). Proposition 3 yields the log-concavity of the rows of this generalization of the Delannoy triangle. (The result stating the log-concavity of the rows of the classical Delannoy triangle appears in Ye [18].) More concretely, Proposition 3 can be applied to the case where  $a = (\dots, 0, 0, 1, b, b^2, \dots)$  and  $q$  itself is given as the convolution

$$q = (\dots, 0, 0, 1, b, b^2, \dots) * (\dots, 0, 0, c, d, 0, 0, \dots)$$

Obviously the  $n^{\text{th}}$  row of this generalized Delannoy triangle is the sequence of coefficients of the monomials of total degree  $n$  in the power series

$$\sum_{m=0}^{\infty} (bx + cy + dxy)^m$$

#### 4. COMPARISON OF METHODS TO PROVE LOG-CONCAVITY

To establish the log-concavity of the rows of a triangular array Kurtz's method [8] assumes a certain weighted recursion relationship to generate from each row the next, the weight sequences themselves being assumed to be convex. On the other hand, Proposition 3 above assumes that the triangular array is a convolution array, generated from log-concave initial side and convolution multiplier sequences.

Kurtz's method works to establish also the log-concavity of rows of certain convolution triangles, which was used in particular to prove log-concavity of  $f$ -vectors of ordinary polytopes (see [12], where convolution triangles have eventually decreasing, not necessarily log-concave initial side sequences). Proposition 3 works also to establish log-concavity of some triangular arrays where a row does not fully determine the next row (classical and generalized Delannoy triangles).

The two approaches have, however, a non-trivial overlap. For example, while an instance of Proposition 3 with convolution multiplier  $q = (\dots, 0, 0, 1, 1, 1 \dots)$ , as appearing implicitly in Hoggar [5], provides a simple approach to verify the log-concavity of  $f$ -vectors of cyclic polytopes, the first proof of this log-concavity property for the larger class of ordinary polytopes was based on an argument amounting to an instance of Kurtz's method [12].

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